Infinite Infrared Regularization in Krein Spaces

Andreas U. Schmidt
aschmidt@math.uni-frankfurt.de
www.math.uni-frankfurt.de/~aschmidt

Fachbereich Mathematik
Johann Wolfgang Goethe–Universität
Frankfurt am Main

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by
Andreas U. Schmidt
Fachbereich Mathematik
Johann Wolfgang Goethe-Universität
60054 Frankfurt am Main, Germany
currently at
Fraunhofer – Institute Secure Telecooperation
Dolivostraße 15
64293 Darmstadt, Germany
Indefiniteness in Quantum Field Theory

The massless scalar field in 1 + 1 dimensions has the two-point function

\[ W_2(x; y) = D_+ (x - y) \]

wherein analytic regularization yields a distribution on \((2\pi)^2\). Upon Fourier transformation it becomes, in lightcone coordinates

\[ p \overset{\text{def}}{=} p_0 + p_1, \]

The tempered distribution

\[ 1 + p_0 + (e^{f}) \overset{\text{def}}{=} \int_1^0 dp e^{f}(p) - e^{f}(0) \]

is not positive definite in general, when \( e^{f}(0) \neq 0 \).
The massless scalar field in 1 + 1 dimensions has the two point function

\[ \mathcal{W}_2(x, y) = -D^+(x - y), \quad -D^+(x) \overset{\text{def}}{=} \lim_{\varepsilon \searrow 0} \frac{1}{4\pi} \ln \left( x^2 + i\varepsilon x^0 \right), \]

wherein analytic regularization yields a distribution on \( \mathcal{S}(\mathbb{R}^2) \).
Indefiniteness in Quantum Field Theory

The massless scalar field in $1 + 1$ dimensions has the two point function

\[ \mathcal{M}_2(x, y) = - D^+(x - y), \quad D^+(x) \equiv \lim_{\varepsilon \searrow 0} \frac{1}{4\pi} \ln (x^2 + i\varepsilon x^0), \]

wherein analytic regularization yields a distribution on $\mathcal{S}(\mathbb{R}^2)$. Upon Fourier transformation it becomes, in lightcone coordinates $p_\pm \equiv p_0 \pm p_1$,

\[ \widetilde{D}^+(p) = \left[ \left( \frac{1}{p_+} \right)_+ \delta(p_-) + \left( \frac{1}{p_-} \right)_+ \delta(p_+) \right] \Theta(p_0). \]
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The tempered distribution

$$\left( \frac{1}{p} \right)_+ \overset{\text{def}}{=} \int_0^\infty dp \frac{\tilde{f}(p) - \tilde{f}(0)}{p}$$

is not positive definite in general, when $\tilde{f}(0) \neq 0$. 
Is Indefiniteness Generic in QFT?

Locality:
Whenever \((x_j - x_{j+1})^2 < 0\), then
\[ W(x_1; \ldots; x_j; x_{j+1}; \ldots; x_n) = W(x_1; \ldots; x_{j+1}; x_j; \ldots; x_n) \]

Positivity:
For \(f; g\), the sesquilinear form
\[ f; g = \sum_{m,n} W_{m+n}(f_m g_n) \]
shall be positive, i.e.,
\[ f; f = 0 \]

Here, \(f_n g_m(x_1; \ldots; x_{m+n}) = f_n(x_1; \ldots; x_n) g_m(x_{n+1}; \ldots; x_{m+n})\), and
\[ f = (f_0; f_1; \ldots) \text{ s.t. } \# f_i f_i = 0, g < 1 \]

Locality and Positivity are in conflict, when trying to construct interacting theories:

Theorem. ([Strocchi 1993])
A local formulation of a field theory with nontrivial gauge invariance, and satisfying all Wightman axioms, contains no charged states in the physical Hilbert space.

The mechanism behind this phenomenon is the presence of virtual states, i.e., physical equivalence classes of the potential, in theories with nontrivial gauge invariance ([Bogolubov et al. 1990], Chapter 10).
Is Indefiniteness Generic in QFT?

Or, is it an artifact of the chosen regularization?
Or, is it an artifact of the chosen regularization? Two fundamental axioms of QFT:

**Locality:** Whenever \((x_j - x_{j+1})^2 < 0\), then
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\mathcal{W}(x_1, \ldots, x_j, x_{j+1}, \ldots, x_n) = \mathcal{W}(x_1, \ldots, x_{j+1}, x_j, \ldots, x_n).
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**Positivity:** For \(f, g \in \mathcal{L}\), the sesquilinear form \((f, g) \equiv \sum_{m,n} \mathcal{M}_{m+n}(f_n^* \otimes g_m)\) shall be positive, i.e., \((f, f) \geq 0\).

Here, \(f_n \otimes g_m(x_1, \ldots, x_{m+n}) \overset{\text{def}}{=} f_n(x_1, \ldots, x_n)g_m(x_{n+1}, \ldots, x_{m+n})\), and \(\mathcal{L} \overset{\text{def}}{=} \{ f = (f_0, f_1, \ldots) \mid f_0 \in \mathbb{C}, f_i \in \mathcal{S}(\mathbb{R}^d), \# \{i, f_i \neq 0\} < \infty \}\).
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**Positivity:** For \(f, g \in \mathcal{S}\), the sesquilinear form \((f, g) \equiv \sum_{m,n} \mathcal{W}_{m+n}(f_n^* \otimes g_m)\) shall be positive, i.e., \((f, f) \geq 0\).

Here, \(f_n \otimes g_m(x_1, \ldots, x_{m+n}) \triangleq f_n(x_1, \ldots, x_n)g_m(x_{n+1}, \ldots, x_{m+n})\), and

\[
\mathcal{S} \equiv \{ f = (f_0, f_1, \ldots) | f_0 \in \mathbb{C}, f_i \in \mathcal{S}(\mathbb{R}^4), \# \{i, f_i \neq 0\} < \infty \}.
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Locality and Positivity are in conflict, when trying to construct interacting theories:

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The mechanism behind this phenomenon is the presence **virtual states**, i.e., physical equivalence classes of the potential, in theories with nontrivial gauge invariance [Bogolubov et al. 1990, Chapter 10].
Physicists Strategy

1. Retain Locality throughout by fixing a local gauge.
2. Perform a functional regularization of emerging singularities by choosing the right test function space.
3. Reconstruct the state space as an infinite inner product space (Wightman reconstruction from the n-point functions.)
4. Try to isolate the negative degrees of freedom.
5. Boldly disregard them as unphysical.

To carry out 3. and 4. one replaces positivity by the weaker Hilbert space structure condition (HS).

Assume that there exist Hilbert seminorms $p_n = (\ldots)$ such that $W_n + m (f_n g_m) p_n (f_n) p_m (g_m)$ for all $f_n \in (\ldots)_n$, $g_m \in (\ldots)_m$. 
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\[ W^n + m(\mathbf{f}_n \mathbf{g}_m) \leq p^n(\mathbf{f}_n) p^m(\mathbf{g}_m) \quad \text{for all } \mathbf{f}_n \in \mathcal{H}^n, \mathbf{g}_m \in \mathcal{H}^m. \]
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\[ p_n = (\ldots)^{1/2} \text{ on } \mathcal{S}(\mathbb{R}^{4n}) \text{ such that } |\mathcal{M}_{n+m}(f^*_n \otimes g_m)| \leq p_n(f_n)p_m(g_m) \]

for all \(f_n \in \mathcal{S}(\mathbb{R}^{4n}), g_m \in \mathcal{S}(\mathbb{R}^{4m}).\)
Towards Krein Spaces

Let $V$ be a vector space equipped with an indefinite inner product $\langle \cdot, \cdot \rangle$. A locally convex topology on $V$ is called majorant, if the inner product $\langle \cdot, \cdot \rangle$ is jointly continuous.


i) To every majorant exists a weaker one defined by a single seminorm.

ii) A seminorm $p$ defines a majorant if $\langle \cdot, \cdot \rangle_p(x) > 0$ for all $x \in V$.

Proposition (B).

If a majorant is normed, i.e., defined by $\| \cdot \| = \langle \cdot, \cdot \rangle^{1/2}$, then exists the metric (Gram) operator $\langle x, y \rangle = \langle x, Jy \rangle$.

Definition (C).

$K$ is a Krein space iff $J$ is completely invertible.

Lemma (D).

A majorant Hilbert topology is maximal, i.e., no weaker majorant exists, iff $J$ has bounded inverse, i.e., $K$ is a Krein space.

A–D render HS effective: They allow the construction of the largest Krein space closure to a given indefinite inner product, and the separation of the negative part, to arrive at a Hilbert space containing all physical information.
Towards Krein Spaces

Let $\mathcal{V}$ be a vector space equipped with an indefinite inner product $\langle . , . \rangle$. A locally convex topology $\tau$ on $\mathcal{V}$ is called majorant, if the inner product $\langle . , . \rangle$ is jointly $\tau$–continuous.

Let $\mathcal{K} \overset{\text{def}}{=} \overline{\mathcal{V}^\tau}$ be the $\tau$–completion of $\mathcal{V}$.
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**Lemma (A).** i) To every majorant exists a weaker one defined by a single seminorm. ii) A seminorm $p$ defines a majorant if $|\langle x , x \rangle| \leq \alpha p(x)^2$, $\alpha > 0$, $\forall x \in \mathcal{V}$. 

**Proposition (B).** If a majorant is normed, i.e., defined by $k : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$

**Definition (C).** $\mathcal{K}$ is a **Krein space** iff $J$ is completely invertible.

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then exists the **metric (Gram) operator**:

$$\langle x , y \rangle = \langle x , Jy \rangle, \quad \forall x , y \in \mathcal{K}.$$
Towards Krein Spaces

Let \( \mathcal{V} \) be a vector space equipped with an indefinite inner product \( \langle . , . \rangle \). A locally convex topology \( \tau \) on \( \mathcal{V} \) is called majorant, if the inner product \( \langle . , . \rangle \) is jointly \( \tau \)-continuous.

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\( \iff \mathcal{K} = \mathcal{K}^+ (\oplus) \mathcal{K}^- , \mathcal{K}^\pm \) complete w.r.t. the weak topology)
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Lemma (D). A majorant Hilbert topology is maximal, i.e., no weaker majorant exists,
iff $J$ has bounded inverse, i.e., $\mathcal{K}$ is a Krein space.
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The Gelfand-Shilov Scheme of Test Function Spaces

Infinite Infrared Regularization in Krein Spaces – p.7/15
\[ \mathcal{I}_\alpha(\mathbb{R}^n) \overset{\text{def}}{=} \left\{ f \in C^\infty(\mathbb{R}^n) \mid \forall k, q \in \mathbb{N}_0: \sup_{x, |\sigma| \leq q} |x^k D_x^\sigma f(x)| \leq C_q A^k k^{k\alpha} \right\}, \]

\[ \mathcal{I}_\beta(\mathbb{R}^n) \overset{\text{def}}{=} \left\{ f \in C^\infty(\mathbb{R}^n) \mid \forall k, q \in \mathbb{N}_0: \sup_{x, |\sigma| \leq q} |x^k D_x^\sigma f(x)| \leq C_k B^q q^{\beta} \right\}, \]

\[ \mathcal{I}_\alpha^\beta(\mathbb{R}^n) \overset{\text{def}}{=} \left\{ f \in C^\infty(\mathbb{R}^n) \mid \forall k, q \in \mathbb{N}_0: \sup_{x, |\sigma| \leq q} |x^k D_x^\sigma f(x)| \leq C A^k B^q k^{k\alpha} q^{\beta} \right\}. \]
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\[ \mathcal{S}_{\alpha\beta}(\mathbb{R}^n) \overset{\text{def}}{=} \left\{ f \in C^\infty(\mathbb{R}^n) \mid \forall k, q \in \mathbb{N}_0: \sup_{x, |\sigma| \leq q} |x^k D^\sigma_x f(x)| \leq C A^k B^q k^{k\alpha} q^{q\beta} \right\}. \]

- Upper index \( \beta \) controls **regularity**, lower index \( \alpha \) **growth**
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\mathcal{S}_\alpha(\mathbb{R}^n) \overset{\text{def}}{=} \left\{ f \in C^\infty(\mathbb{R}^n) \mid \forall k, q \in \mathbb{N}_0 : \sup_{x, |\sigma| \leq q} |x^k D_\sigma^\alpha f(x)| \leq C_q A^k k^{k\alpha} \right\},
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\]

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- Upper index $\beta$ controls regularity, lower index $\alpha$ growth
- Spaces with $\beta = 1$ contain only real analytic functions, for $\beta < 1$ only entire fn’s
The Gelfand-Shilov Scheme of Test Function Spaces

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- Spaces with \( \beta = 1 \) contain only **real analytic** functions, for \( \beta < 1 \) only **entire** fn’s
- **Fourier transformation** exchanges indices \( \mathcal{F}: \mathcal{I}_\beta \sim \mathcal{I}_\alpha \)
- Limits: \( \mathcal{I}_\alpha \cong \lim_{\beta \to \infty} \mathcal{I}_\beta \) and \( \mathcal{I}_\beta \cong \lim_{\alpha \to \infty} \mathcal{I}_\alpha \)
- Old acquaintance: **Schwartz space** \( \mathcal{S} \cong \lim_{\alpha, \beta \to \infty} \mathcal{I}_\alpha \);
  
  **Compactly supported** test functions \( \mathcal{D} \cong \mathcal{I}_0(\mathbb{R}^n) \);
  Test functions for **Fourier hyperfunctions** \( \mathcal{P}_* \cong \mathcal{I}_1^1 \).
Strength of Singularity: Examples

The two-point function of the 1 + 1-dimensional dipole field, solution of $\Box^2 = 0$, is in $0$, for $\theta_1 = 2$.

The vertex operator $\exp(i g')$ of this field is in $0$, for $\theta_1 = 4$.

In the temporal variables, $\exp(i g')$ for the dipole field in 3 + 1 dimensions is in $0 + 3 = 2$, $\theta_1 = 2$. 

Suitable test function spaces for certain QFT models [Moschella/Strocchi 1992].
The two-point function of the $1 + 1$-dimensional dipole field, solution of $\Box^2 \varphi = 0$, is in $\mathcal{S}_\alpha'$, for $\alpha < 1/2$.

\[ \nabla \text{The vertex operator } : \exp(i g \varphi) : \text{ of this field is in } \mathcal{S}_\alpha', \text{ for } \alpha < 1/4. \]

\[ \square \text{ In the temporal variables, } : \exp(i g \varphi) : \text{ for the dipole field in } 3 + 1 \text{ dimensions is in } \mathcal{S}_\alpha^{\beta'}, \alpha + \beta < 3/2, \alpha < 1/2. \]
A Model with Infinite Infrared Singularity
A Model with Infinite Infrared Singularity

\[ \langle f, g \rangle \overset{\text{def}}{=} (f, g)_{L^2} - \sum_{k=0}^{\infty} c_k^2 f^{(k)}(0) g^{(k)}(0), \quad c_k \in \mathbb{R} \setminus \{0\}, \]

is an indefinite product,
A Model with Infinite Infrared Singularity

\[ \langle f, g \rangle \overset{\text{def}}{=} (f, g)_{L^2} - \sum_{k=0}^{\infty} c_k^2 f^{(k)}(0) g^{(k)}(0), \quad c_k \in \mathbb{R} \setminus \{0\}, \]

is an indefinite product, characterized by the infra-exponential symbol of order \(1/(2\delta)\)

\[ J(\xi) \overset{\text{def}}{=} \sum_{k=0}^{\infty} c_k^2 \xi^k, \quad |J(\xi)| \leq C_\varepsilon e^{\varepsilon|\xi|^{1/(2\delta)}}, \quad \forall \varepsilon > 0. \]
A Model with Infinite Infrared Singularity

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\langle f, g \rangle \overset{\text{def}}{=} (f, g)_{L^2} - \sum_{k=0}^{\infty} c_k^2 f^{(k)}(0) g^{(k)}(0), \quad c_k \in \mathbb{R} \setminus \{0\},
\]

is an **indefinite product**, characterized by the **infra-exponential symbol of order** \(1/(2\delta)\)

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J(\xi) \overset{\text{def}}{=} \sum_{k=0}^{\infty} c_k^2 \xi^k, \quad |J(\xi)| \leq C_\varepsilon e^{\varepsilon |\xi|^{1/(2\delta)}}, \quad \forall \varepsilon > 0.
\]

It is well defined on \(\mathcal{S}_\alpha^\beta(\mathbb{R})\) for any \(\alpha\) and \(\beta < \delta\) \((\beta \leq \delta \text{ suffices})\), since then

\[
|c_k^2| \leq C \frac{\theta^k}{D^k e^{2k\delta} k^{2k\delta}}, \quad \forall D > 0 \ \exists \theta \in (0, 1), \ C > 0,
\]

\[
|f^{(k)}(0)| \leq C_f (B + \rho)^k k^{k\beta}, \text{ some } B, C_f > 0, \quad \forall \rho > 0.
\]
A Model with Infinite Infrared Singularity

\[ \langle f, g \rangle \overset{\text{def}}{=} (f, g)_{L^2} - \sum_{k=0}^{\infty} c_k^{2} f^{(k)}(0) g^{(k)}(0) , \quad c_k \in \mathbb{R} \setminus \{0\} , \]

is an indefinite product, characterized by the infra-exponential symbol of order \( 1/(2\delta) \)

\[ J(\xi) \overset{\text{def}}{=} \sum_{k=0}^{\infty} c_k^{2} \xi^k , \quad |J(\xi)| \leq C_{\varepsilon} \varepsilon^{\varepsilon |\xi|^{1/(2\delta)}} , \quad \forall \varepsilon > 0 . \]

It is well defined on \( \mathcal{S}_\alpha^\beta(\mathbb{R}) \) for any \( \alpha \) and \( \beta < \delta \) (\( \beta \leq \delta \) suffices), since then

\[ |c_k^2| \leq C \frac{\theta_k}{D_k e^{2k\delta} k^{2k\delta}} , \quad \forall D > 0 \exists \theta \in (0, 1) , C > 0 , \]

\[ |f^{(k)}(0)| \leq C_f (B + \rho)^k k^{k\beta} , \text{ some } B, C_f > 0 , \quad \forall \rho > 0 . \]

**Task: Infinite Infrared Regularization**

- Construct the maximal Krein space completion of \( \{ \mathcal{S}_\alpha^{\beta<\delta} , \langle \cdot , \cdot \rangle \} ; \)
A Model with Infinite Infrared Singularity

\[ \langle f, g \rangle \overset{\text{def}}{=} (f, g)_{L^2} - \sum_{k=0}^{\infty} c_k^2 f^{(k)}(0) g^{(k)}(0), \quad c_k \in \mathbb{R} \setminus \{0\}, \]

is an \textit{indefinite product}, characterized by the \textit{infra-exponential symbol of order} \( 1/(2\delta) \)

\[ J(\xi) \overset{\text{def}}{=} \sum_{k=0}^{\infty} c_k^2 \xi^k, \quad |J(\xi)| \leq C e^{\epsilon |\xi|^{1/(2\delta)}}, \quad \forall \epsilon > 0. \]

It is well defined on \( \mathcal{S}^\beta_\alpha(\mathbb{R}) \) for any \( \alpha \) and \( \beta < \delta \) (\( \beta \leq \delta \) suffices), since then

\[ |c_k^2| \leq C \frac{\theta^k}{D^k e^{2k\delta} k^{2k\delta}}, \quad \forall D > 0 \exists \theta \in (0, 1), C > 0, \]

\[ |f^{(k)}(0)| \leq C_f (B + \rho)^k k^{k \beta}, \text{ some } B, C_f > 0, \quad \forall \rho > 0. \]

**Task: Infinite Infrared Regularization**

- Construct the maximal Krein space completion of \( \{ \mathcal{S}^\beta_\alpha, \langle \cdot, \cdot \rangle \} \);

- Identify its largest, positive definite, subspace,
A Model with Infinite Infrared Singularity

\[ \langle f , g \rangle \overset{\text{def}}{=} (f , g)_{L^2} - \sum_{k=0}^{\infty} c_k^2 f^{(k)}(0) g^{(k)}(0), \quad c_k \in \mathbb{R} \setminus \{0\}, \]

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It is well defined on \(\mathcal{S}_{\alpha}^{\beta}(\mathbb{R})\) for any \(\alpha\) and \(\beta < \delta\) (\(\beta \leq \delta\) suffices), since then

\[ |c_k^2| \leq C \frac{\theta^k}{D e^{2k\delta \varepsilon^2 k^2 \delta}}, \quad \forall D > 0 \ \exists \theta \in (0, 1), \ C > 0, \]

\[ |f^{(k)}(0)| \leq C_f (B + \rho)^k k!^\beta, \text{ some } B, C_f > 0, \quad \forall \rho > 0. \]

**Task: Infinite Infrared Regularization**

- Construct the maximal Krein space completion of \(\{\mathcal{S}_{\alpha}^{\beta < \delta}, \langle . , . \rangle\}\);
- Identify its largest, positive definite, subspace,
- which is invariant under the Heisenberg algebra \(\langle \hat{q} \overset{\text{def}}{=} x, \quad \hat{p} \overset{\text{def}}{=} -i \frac{d}{dx} \rangle\).
A Model with Infinite Infrared Singularity

\[ \langle f, g \rangle \overset{\text{def}}{=} (f, g)_{L^2} - \sum_{k=0}^{\infty} c_k f^{(k)}(0) g^{(k)}(0), \quad c_k \in \mathbb{R} \setminus \{0\}, \]

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\[ |c_k^2| \leq C \frac{\theta^k}{D^k e^{2k\delta} k^{2k\delta}}, \quad \forall D > 0 \exists \theta \in (0, 1), C > 0, \]

\[ |f^{(k)}(0)| \leq C_f (B + \rho)^k k^{k\beta}, \text{ some } B, C_f > 0, \quad \forall \rho > 0. \]

**Task: Infinite Infrared Regularization**

- Construct the maximal Krein space completion of \( \{ \mathcal{S}_\alpha^{\beta<\delta}, \langle \cdot , \cdot \rangle \} \);
- Identify its largest, positive definite, subspace,
- which is invariant under the Heisenberg algebra \( \langle \hat{q} \overset{\text{def}}{=} x \cdot, \quad \hat{p} \overset{\text{def}}{=} -i \frac{d}{dx} \rangle \).

Heuristics:

- The singularity outside the origin matters not for the regularization.
A Model with Infinite Infrared Singularity

\[ \langle f, g \rangle \overset{\text{def}}{=} (f, g)_{L^2} - \sum_{k=0}^{\infty} c_k^2 f^{(k)}(0) g^{(k)}(0), \quad c_k \in \mathbb{R} \setminus \{0\}, \]

is an indefinite product, characterized by the infra-exponential symbol of order \( 1/(2\delta) \)

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It is well defined on \( \mathcal{H}_\alpha^\beta(\mathbb{R}) \) for any \( \alpha \) and \( \beta < \delta \) (\( \beta \leq \delta \) suffices), since then

\[ |c_k^2| \leq C \frac{\theta^k}{D^k e^{2k\delta} k_{2k\delta}}, \quad \forall D > 0 \exists \theta \in (0, 1), C > 0, \]

\[ |f^{(k)}(0)| \leq C_f (B + \rho)^k k_{k\beta}, \quad \text{some } B, C_f > 0, \quad \forall \rho > 0. \]

**Task: Infinite Infrared Regularization**

- Construct the maximal Krein space completion of \( \{ \mathcal{H}_\alpha^\beta, \langle \cdot, \cdot \rangle \} \);
- Identify its largest, positive definite, subspace,
- which is invariant under the Heisenberg algebra \( \langle \hat{q} \overset{\text{def}}{=} x, \hat{p} \overset{\text{def}}{=} -i\frac{d}{dx} \rangle \).

**Heuristics:**

- The singularity outside the origin matters not for the regularization.
- The Heisenberg algebra is ubiquitous as an observable algebra.
Neutral Decomposition

Set $k$ and construct neutral functions $N$ defined as $\text{span } f_k g_k = 2^0$ satisfying

(i) $L_2 = \sum_k 2^k$

(ii) $f_k(i_0) = i_k$

(iii) $(k; l) L_2 = 0$ for $k \leq 6$ and $l \geq 0$

(iv) $h_i = 0$ for $i \geq 0$

The decomposition $f = f_N + N \sum_i f_i i$, with $f_i(i_0) = i_{i k}$, defines a majorant on $f$ through the seminorm $p(f) = \lim N!L(h f_i + f_N + N \sum_i n f_i)$. In $K$, this gives rise to a Hilbert scalar product on $f$ with $k$, defined as $p(f; g) = \lim N!L(f; g)_{N!X}$. Where $f + \sum_i f_i g_i = \lim N!f N + \sum_i n f_i$ (convergent in $K$).
Neutral Decomposition

Set \( \gamma_k \overset{\text{def}}{=} k^k \delta \) and construct neutral functions \( N \overset{\text{def}}{=} \text{span}\{\chi_k\}_{k \in \mathbb{N}_0} \subset H^{\beta}_\alpha \) satisfying

\begin{align*}
\text{i)} & \quad L^2 = c^2_k \quad \text{for all } k \\
\text{ii)} & \quad (i_k)(0) = i_k \\
\text{iii)} & \quad (\gamma_k; l)_{L^2} = 0 \quad \forall k, l \\
\text{iv)} & \quad h_{i_k}(0) = 0 \quad \forall i_k; l
\end{align*}
Neutral Decomposition

Set $\gamma_k \overset{\text{def}}{=} k^\delta$ and construct neutral functions $\mathcal{N} \overset{\text{def}}{=} \text{span}\{\chi_k\}_{k \in \mathbb{N}_0} \subset \mathcal{S}_\alpha^\beta$ satisfying

i) $\|\chi_k\|_{L^2}^2 = c_k^2 \gamma_k^2$
Neutral Decomposition

Set $\gamma_k \overset{\text{def}}{=} k^\delta$ and construct neutral functions $\mathcal{N} \overset{\text{def}}{=} \text{span}\{\chi_k\}_{k \in \mathbb{N}_0} \subset \mathcal{H}_\alpha$ satisfying

i) $\|\chi_k\|_{L^2}^2 = c_k^2 \gamma_k^2$ \hspace{1cm} ii) $\chi_k^{(i)}(0) = \delta_{ik} \cdot \gamma_k$


Neutral Decomposition

Set \( \gamma_k \overset{\text{def}}{=} k \delta \) and construct neutral functions \( \mathcal{N} \overset{\text{def}}{=} \text{span}\{\chi_k\}_{k \in \mathbb{N}_0} \subset \mathcal{H}_\alpha^\beta \) satisfying

i) \( \|\chi_k\|_{L^2}^2 = c_k^2 \gamma_k^2 \)

ii) \( \chi_k^{(i)}(0) = \delta_{ik} \cdot \gamma_k \)

iii) \( (\chi_k, \chi_l)_{L^2} = 0, \forall k \neq l \)
Neutral Decomposition

Set \( \gamma_k \overset{\text{def}}{=} k \delta \) and construct neutral functions \( \mathcal{N} \overset{\text{def}}{=} \text{span}\{\chi_k\}_{k \in \mathbb{N}_0} \subset \mathcal{H}_\alpha^\beta \) satisfying

i) \( \|\chi_k\|_{L^2}^2 = c_k^2 \gamma_k^2 \)

ii) \( \chi_k^{(i)}(0) = \delta_{ik} \cdot \gamma_k \)

iii) \( (\chi_k, \chi_l)_{L^2} = 0, \forall k \neq l \)

\( \implies \)

iv) \( \langle \chi_k, \chi_l \rangle = 0, \forall k, l \)
Neutral Decomposition

Set $\gamma_k \overset{\text{def}}{=} k^k \delta$ and construct neutral functions $\mathcal{N} \overset{\text{def}}{=} \text{span}\{\chi_k\}_{k \in \mathbb{N}_0} \subset \mathcal{H}_\alpha^\beta$ satisfying

i) $\|\chi_k\|_{L^2}^2 = c_k^2 \gamma_k^2$  

ii) $\chi_k^{(i)}(0) = \delta_{ik} \cdot \gamma_k$

iii) $(\chi_k, \chi_l)_{L^2} = 0, \forall k \neq l$  \implies iv) $(\chi_k, \chi_l) = 0, \forall k, l$

The decomposition $f = f^{N+} + \sum_{i=0}^{N} f^i \chi_i$, with $f^i = \frac{f^{(i)}(0)}{\gamma_i}$,
Neutral Decomposition

Set $\gamma_k \overset{\text{def}}{=} k^k \delta$ and construct neutral functions $\mathcal{N} \overset{\text{def}}{=} \text{span}\{\chi_k\}_{k \in \mathbb{N}_0} \subset \mathcal{S}_\alpha^\beta$ satisfying

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ii) $\chi_k^{(i)}(0) = \delta_{ik} \cdot \gamma_k$

iii) $(\chi_k, \chi_l)_{L^2} = 0$, $\forall k \neq l$ \implies iv) $(\chi_k, \chi_l) = 0$, $\forall k, l$

The decomposition $f = f^{N+} + \sum_{i=0}^{N} f^i \chi_i$, with $f^i = \frac{f^{(i)}(0)}{\gamma_i}$

defines a majorant $\tau$ on $\{\mathcal{S}_\alpha^\beta < \delta, \langle ., . \rangle\}$ through the seminorm

$$p(f)^2 \overset{\text{def}}{=} \lim_{N \to \infty} \left[ \langle f^{N+}, f^{N+} \rangle + \sum_{i=0}^{N} \left\{ |\langle f, \chi_i \rangle|^2 + |f^i|^2 \right\} \right].$$

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Neutral Decomposition

Set $\gamma_k \overset{\text{def}}{=} k^k \delta$ and construct neutral functions $N \overset{\text{def}}{=} \text{span}\{\chi_k\}_{k \in \mathbb{N}_0} \subset \mathcal{H}_\alpha^\beta$ satisfying

i) $\|\chi_k\|^2_{L^2} = c_k^2 \gamma_k^2$

ii) $\chi_k^{(i)}(0) = \delta_{ik} \cdot \gamma_k$

iii) $\langle \chi_k, \chi_l \rangle_{L^2} = 0, \ \forall k \neq l \quad \Longrightarrow \quad$ iv) $\langle \chi_k, \chi_l \rangle = 0, \ \forall k, l$

The decomposition $f = f^{N+} + \sum_{i=0}^N f^i \chi_i$, with $f^i = \frac{f^{(i)}(0)}{\gamma_i}$, defines a majorant $\tau$ on $\{\mathcal{H}_\alpha^\beta<\delta, \langle . , . \rangle\}$ through the seminorm

$$p(f)^2 \overset{\text{def}}{=} \lim_{N \to \infty} \left[ \langle f^{N+} , f^{N+} \rangle + \sum_{i=0}^N \left\{ |\langle f , \chi_i \rangle|^2 + |f^i|^2 \right\} \right].$$

In $\mathcal{K} \overset{\text{def}}{=} \mathcal{H}_\alpha^\beta^\top$, this gives rise to a Hilbert scalar product $(.,.)$ with $\| . \| \overset{\text{def}}{=} p(.) = (.,.)^{1/2}$
Neutral Decomposition

Set \( \gamma_k \overset{\text{def}}{=} k \cdot k \delta \) and construct neutral functions \( \mathcal{N} \overset{\text{def}}{=} \text{span}\{\chi_k\}_{k \in \mathbb{N}_0} \subset \mathcal{I}_\alpha^\beta \) satisfying

1) \( \|\chi_k\|_{L^2}^2 = c_k^2 \gamma_k^2 \)
2) \( \chi_k^{(i)}(0) = \delta_{ik} \cdot \gamma_k \)
3) \( \langle \chi_k, \chi_l \rangle_{L^2} = 0, \forall k \neq l \) \implies
4) \( \langle \chi_k, \chi_l \rangle = 0, \forall k, l \)

The decomposition \( f = f^{N^+} + \sum_{i=0}^{N} f^i \chi_i \), with \( f^i = \frac{f^{(i)}(0)}{\gamma_i} \),
defines a majorant \( \tau \) on \( \{\mathcal{I}_\alpha^{\beta < \delta}, \langle . , . \rangle\} \) through the seminorm

\[
p(f)^2 \overset{\text{def}}{=} \lim_{N \to \infty} \left[ \langle f^{N^+}, f^{N^+} \rangle + \sum_{i=0}^{N} \left\{ |\langle f, \chi_i \rangle|^2 + |f^i|^2 \right\} \right].
\]

In \( \mathcal{K} \overset{\text{def}}{=} \mathcal{I}_\alpha^{\beta^\top} \), this gives rise to a Hilbert scalar product \( \langle . , . \rangle \) with \( \| . \| \overset{\text{def}}{=} p(.) = ( . , . )^{1/2} \)

\[
\langle f, g \rangle \overset{\text{def}}{=} \langle f^+, g^+ \rangle + \sum_{i=0}^{\infty} \left\{ \langle f, \chi_i \rangle \langle \chi_i, g \rangle + \bar{f^i} \bar{g^i} \right\}, \quad \forall f, g \in \mathcal{K},
\]

where \( f^+ \overset{\text{def}}{=} \lim_{N \to \infty} f^{N^+} \) (convergent in \( \mathcal{K} \)).
\[ \left\| \sum_{i=0}^{\infty} f^i \chi_i \right\|_{L^2}^2 \leq \sum_{i=0}^{\infty} \left\| f^i \chi_i \right\|_{L^2}^2 \]
\[
\left\| \sum_{i=0}^{\infty} f^i \chi_i \right\|_{L^2}^2 \leq \sum_{i=0}^{\infty} |f^{(i)}(0) c_i|^2
\]
\[ \sum_{i=0}^{\infty} f^i X_i \leq \sum_{i=0}^{\infty} \left| CC_f \left( \frac{\theta(B + \rho)}{De^{2\delta}} \right)^i \right|^{2i} e^{2i(\delta - \beta)} \]
Finiteness of $\| \cdot \|$ \[\begin{align*}
\| \sum_{i=0}^{\infty} f^i \chi_i \|_{L^2}^2 &< \infty
\end{align*}\]
\[ \left\| \sum_{i=0}^{\infty} f^i \chi_i \right\|_{L^2}^2 < \infty \text{ and } f \in L^2 \implies f^{N+} \xrightarrow{N \to \infty} f^+ \in L^2 \]
\[
\left\| \sum_{i=0}^{\infty} f^i \chi_i \right\|_{L^2}^2 < \infty \quad \text{and} \quad f \in L^2 \implies f^N \xrightarrow{N \to \infty} f^+ \in L^2
\]

\[
\implies \lim_{N \to \infty} \left\langle f^{N+}, f^{N+} \right\rangle = \|f^+\|_{L^2}
\]
\[
\sum_{i=0}^{\infty} f^i \chi_i \|_{L^2}^2 < \infty \quad \text{and} \quad f \in L^2 \implies f^N+ \xrightarrow{N \to \infty} f+ \in L^2
\]

\[
\implies \lim_{N \to \infty} \langle f^N+, f^N+ \rangle < \infty.
\]
\[ \left\| \sum_{i=0}^{\infty} f^{i} \chi_{i} \right\|_{L^2}^2 < \infty \quad \text{and} \quad f \in L^2 \implies f^N \xrightarrow{N \to \infty} f^+ \in L^2 \]

\[ \implies \lim_{N \to \infty} \langle f^N^+, f^N^+ \rangle < \infty. \]

\[ |\langle f, \chi_i \rangle|^2 + |f^i|^2 = |(f, \chi_i)_{L^2} - \sum_{k=0}^{\infty} c_k f^{(k)}(0) \chi^{(k)}_{i}(0)|^2 + |f^i|^2 \]
\[
\left\| \sum_{i=0}^{\infty} f^i \chi_i \right\|_{L^2}^2 < \infty \quad \text{and} \quad f \in L^2 \implies f^N_+ \xrightarrow{N \to \infty} f^+ \in L^2
\]

\[
\implies \lim_{N \to \infty} \langle f^N_+ , f^N_+ \rangle < \infty.
\]

\[
|\langle f , \chi_i \rangle|^2 + |f^i|^2 \leq |(f , \chi_i)_{L^2}|^2 + c_i \gamma_i^2 |f^{(i)}(0)|^2 + \frac{|f^{(i)}(0)|^2}{\gamma_i^2}.
\]
\[ \left\| \sum_{i=0}^{\infty} f^i \chi_i \right\|_{L^2}^2 < \infty \quad \text{and} \quad f \in L^2 \implies f^{N+} \xrightarrow{N \to \infty} f^+ \in L^2 \]

\[ \implies \lim_{N \to \infty} \langle f^{N+}, f^{N+} \rangle < \infty. \]

\[ |\langle f, \chi_i \rangle|^2 + |f^i|^2 \leq |(f, \chi_i)_{L^2}|^2 + c_i^4 \gamma_i^2 |f^{(i)}(0)|^2 + \frac{|f^{(i)}(0)|^2}{\gamma_i^2}. \]

\[ |(f, \chi_i)_{L^2}|^2 \leq \|f\|_{L^2}^2 c_i^2 \gamma_i^2 \]
\[ \sum_{i=0}^{\infty} f^i \chi_i \|_{L^2}^2 < \infty \text{ and } f \in L^2 \implies f^N \xrightarrow{N \to \infty} f^+ \in L^2 \]

\[ \implies \lim_{N \to \infty} \langle f^N^+, f^N^+ \rangle < \infty. \]

\[ |\langle f, \chi_i \rangle|^2 + |f^i|^2 \leq |(f, \chi_i)_{L^2}|^2 + c_i^4 \gamma_i^2 |f^{(i)}(0)|^2 + \frac{|f^{(i)}(0)|^2}{\gamma_i^2}. \]

\[ |(f, \chi_i)_{L^2}|^2 \leq \|f\|_{L^2}^2 c_i^2 \gamma_i^2 \leq \|f\|_{L^2}^2 \frac{C \theta^i}{D_i e^{2i\delta}}. \]
\[ \left\| \sum_{i=0}^{\infty} f^i \chi_i \right\|_{L^2}^2 < \infty \text{ and } f \in L^2 \implies f^N \xrightarrow{N \to \infty} f^+ \in L^2 \]

\[ \implies \lim_{N \to \infty} \langle f^N^+, f^N^+ \rangle < \infty. \]

\[ |\langle f, \chi_i \rangle|^2 + |f^i|^2 \leq |\langle f, \chi_i \rangle|_{L^2}^2 + c_i^4 \gamma_i^2 |f^{(i)}(0)|^2 + \frac{|f^{(i)}(0)|^2}{\gamma_i^2}. \]

\[ |\langle f, \chi_i \rangle|_{L^2}^2 \leq \|f\|_{L^2}^2 c_i^2 \gamma_i^2 \leq \|f\|_{L^2}^2 \frac{C \theta^i}{D^i e^{2i\delta}}. \]

\[ c_i^4 \gamma_i^2 |f^{(i)}(0)|^2 \leq \frac{C^2 C_f^2}{i^{2i(\delta - \beta)}} \left( \frac{\theta(B + \rho)}{e^{2\delta} D} \right)^{2i}. \]
\[ \left\| \sum_{i=0}^{\infty} f^i \chi_i \right\|^2_{L^2} < \infty \text{ and } f \in L^2 \implies f^{N+} \xrightarrow{N \to \infty} f^+ \in L^2 \]

\[ = \lim_{N \to \infty} \left\langle f^{N+}, f^{N+} \right\rangle < \infty. \]

\[ |\langle f, \chi_i \rangle|^2 + |f^i|^2 \leq |\langle f, \chi_i \rangle|_{L^2}^2 + c_i^4 \gamma_i^2 |f^{(i)}(0)|^2 + \frac{|f^{(i)}(0)|^2}{\gamma_i^2}. \]

\[ |\langle f, \chi_i \rangle|_{L^2}^2 \leq \|f\|^2_{L^2} c_i^2 \gamma_i^2 \leq \|f\|^2_{L^2} \frac{C \theta^i}{D_i e^{2i\delta}}. \]

\[ c_i^4 \gamma_i^2 |f^{(i)}(0)|^2 \leq \frac{C^2 C_f^2}{i^{2i(\delta-\beta)}} \left( \frac{\theta(B + \rho)}{e^{2\delta} D} \right)^{2i}. \]

\[ \frac{|f^{(i)}(0)|^2}{\gamma_i^2} \leq \frac{C_f^2}{i^{2i(\delta-\beta)}} (B + \rho)^{2i}. \]
Finiteness of $\| \|$

$$\left\| \sum_{i=0}^{\infty} f^i \chi_i \right\|_{L^2}^2 < \infty \quad \text{and} \quad f \in L^2 \implies f^{N+} \xrightarrow{N \to \infty} f^+ \in L^2$$

$$\implies \lim_{N \to \infty} \langle f^{N+}, f^{N+} \rangle < \infty.$$ 

$$|\langle f, \chi_i \rangle|^2 + |f^i|^2 \leq |(f, \chi_i)_{L^2}|^2 + c_i^4 \gamma_i^2 |f^{(i)}(0)|^2 + \frac{|f^{(i)}(0)|^2}{\gamma_i^2}.$$ 

$$|(f, \chi_i)_{L^2}|^2 \leq \|f\|_{L^2}^2 c_i^2 \gamma_i^2 \leq \|f\|_{L^2}^2 \frac{C \theta^i}{D_i e^{2i\delta}}.$$ 

$$c_i^4 \gamma_i^2 |f^{(i)}(0)|^2 \leq \frac{C^2 C_f^2}{i^2 (\delta - \beta)} \left( \frac{\theta (B + \rho)}{e^{2\delta} D} \right)^{2i}.$$ 

$$\frac{|f^{(i)}(0)|^2}{\gamma_i^2} \leq \frac{C_f^2}{i^2 (\delta - \beta)} (B + \rho)^{2i}.$$ 

$$\implies \sum_{i=0}^{\infty} \left\{ |\langle f, \chi_i \rangle|^2 + |f^i|^2 \right\} < \infty.$$
\[ \left\| \sum_{i=0}^{\infty} f^i \chi_i \right\|_{L^2}^2 < \infty \quad \text{and} \quad f \in L^2 \implies f^+ \xrightarrow{N \to \infty} f^+ \in L^2 \]

\[ \implies \lim_{N \to \infty} \langle f^N, f^N \rangle < \infty. \]

\[ |\langle f, \chi_i \rangle|^2 + |f^i|^2 \leq |(f, \chi_i)_{L^2}|^2 + c_i^4 \gamma_i^2 |f^{(i)}(0)|^2 + \frac{|f^{(i)}(0)|^2}{\gamma_i^2}. \]

\[ |(f, \chi_i)_{L^2}|^2 \leq \|f\|_{L^2}^2 \frac{C\theta^i}{D^i e^{2i\delta}}. \]

\[ c_i^4 \gamma_i^2 |f^{(i)}(0)|^2 \leq \frac{C_f^2 C_f^2}{i^2 (\delta - \beta)} \left( \frac{\theta (B + \rho)}{e^{2\delta} D} \right)^{2i}. \]

\[ \frac{|f^{(i)}(0)|^2}{\gamma_i^2} \leq \frac{C_f^2}{i^2 (\delta - \beta)} (B + \rho)^{2i}. \]

\[ \implies \sum_{i=0}^{\infty} \left\{ |\langle f, \chi_i \rangle|^2 + |f^i|^2 \right\} < \infty \implies p(f)^2 = \|f\|^2 < \infty. \]
Separation and Decomposition

Set \( P \) defined as \( f^2(f(0) = 0; 8k \geq 0). \) On \( P \), \( h: i = (i) \) \( L^2 \) is positive definite.

**Lemma.** The mapping \( P: ! K; f \rightarrow f + \) is continuous in the topology, has a continuous extension to \( K \), and is an \( (i) \)–orthogonal projection onto \( P \), which induces the decomposition \( K = P N \).

But: \( P \) is too big (it contains neutral elements). Further decomposition is necessary.

**Lemma.** The functionals \( F_i(f) \) defined as \( h_i; f_i, f^2 \), are \( k: k \)–bounded, and thus possess unique continuations to \( K \). The vector representatives \( v_i \in K, F_i((i)) = (v_i; : \) are contained in \( P \). They satisfy

1. \( h_i; v_i = (v_i; v_i) = 1 \)
2. \( (v_i; v_j) = h_i; v_j = 0, i \neq j \)
3. \( h_v_i; v_i = 0 \)
4. \( h_v_i; f_i = f_i, \) for all \( f \in \).

The \( v_i \) span a closed Hilbert subspace \( N_i \) of \( K \), isomorphic to the \( (i) \)–dual of \( N \).

**Lemma.** Set \( H \) defined as \( P + \) where the topology \( + \) is induced by the quadratic Hilbert norm \( p^+((i)) \) defined as \( h : i \) on \( P \). Then holds the \( (i) \)–orthogonal decomposition \( K = H N \).
Set $\mathcal{P} \overset{\text{def}}{=} \{ f \in \mathcal{L}_\alpha^\beta \mid f^{(k)}(0) = 0, \forall k \in \mathbb{N}_0 \}$. 
Separation and Decomposition

Set \( \mathcal{P} \) def \( \{ f \in \mathcal{L}_\alpha^\beta \mid f^{(k)}(0) = 0, \ \forall k \in \mathbb{N}_0 \} \). On \( \mathcal{P}, \langle ., . \rangle = ( ., . )_{L^2} \) is positive definite.
Set $\mathcal{P} \overset{\text{def}}{=} \{ f \in \mathcal{S}^\beta_{\alpha} \mid f^{(k)}(0) = 0, \ \forall k \in \mathbb{N}_0 \}$. On $\mathcal{P}$, $\langle \cdot, \cdot \rangle = (\cdot, \cdot)_{L^2}$ is positive definite.

**Lemma.** The mapping $P : \mathcal{S}^\beta_{\alpha} \longrightarrow \mathcal{K}; \ f \longmapsto f^+$ is continuous in the topology $\tau$, has a continuous extension to $\mathcal{K}$, and is an $(\cdot, \cdot)$–orthogonal projection onto $\overline{\mathcal{P}}^\tau$, which induces the decomposition $\mathcal{K} = \overline{\mathcal{P}}^\tau \oplus \overline{\mathcal{N}}^\tau$. 
Set $\mathcal{P} \overset{\text{def}}{=} \{ f \in \mathcal{I}^{\beta}_\alpha \mid f^{(k)}(0) = 0, \forall k \in \mathbb{N}_0 \}$. On $\mathcal{P}$, $\langle . , . \rangle = ( . , . )_{L^2}$ is positive definite.

**Lemma.** The mapping $P : \mathcal{I}^{\beta}_\alpha \longrightarrow \mathcal{K}$; $f \longmapsto f^+$ is continuous in the topology $\tau$, has a continuous extension to $\mathcal{K}$, and is an $( . , . )$–orthogonal projection onto $\overline{\mathcal{P}}^\tau$, which induces the decomposition $\mathcal{K} = \overline{\mathcal{P}}^\tau \oplus \overline{\mathcal{N}}^\tau$.

**But:** $\overline{\mathcal{P}}^\tau$ is too big (it contains neutral elements). Further decomposition is necessary.
Set $\mathcal{P} \overset{\text{def}}{=} \{ f \in \mathcal{L}_\alpha^\beta \mid f^{(k)}(0) = 0, \forall k \in \mathbb{N}_0 \}$. On $\mathcal{P}$, $\langle . , . \rangle = ( . , . )_{L_2}$ is positive definite.

**Lemma.** The mapping $P: \mathcal{L}_\alpha^\beta \rightarrow \mathcal{K}; \ f \mapsto f^+$ is continuous in the topology $\tau$, has a continuous extension to $\mathcal{K}$, and is an $(. , . )$–orthogonal projection onto $\overline{\mathcal{P}^\tau}$, which induces the decomposition $\mathcal{K} = \overline{\mathcal{P}^\tau} \oplus \overline{\mathcal{N}^\tau}$.

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**Lemma.** The functionals $F_i( f) = \langle \chi_i , f \rangle$, $f \in \mathcal{L}_\alpha^\beta$, are $\| . \| –$bounded, and thus possess unique continuations to $\mathcal{K}$. The vector representatives $v_i \in \mathcal{K}$, $F_i( . ) = (v_i , . )$ are contained in $\overline{\mathcal{P}^\tau}$. 
Separation and Decomposition

Set \( \mathcal{P} \overset{\text{def}}{=} \{ f \in \mathcal{L}_\alpha^\beta \mid f^{(k)}(0) = 0, \forall k \in \mathbb{N}_0 \} \). On \( \mathcal{P} \), \( \langle ., . \rangle = (., .)_{L_2} \) is positive definite.

**Lemma.** The mapping \( P : \mathcal{L}_\alpha^\beta \rightarrow \mathcal{K} ; \ f \mapsto f^+ \) is continuous in the topology \( \tau \), has a continuous extension to \( \mathcal{K} \), and is an \( (., .) \)–orthogonal projection onto \( \overline{\mathcal{P}}^\tau \), which induces the decomposition \( \mathcal{K} = \overline{\mathcal{P}}^\tau \oplus \overline{\mathcal{N}}^\tau \).

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\[
i) \ (\langle \chi_i, v_i \rangle = (v_i, v_i) = 1
\]
Set $\mathcal{P} \overset{\text{def}}{=} \{ f \in \mathcal{L}_\alpha^\beta \mid f^{(k)}(0) = 0, \forall k \in \mathbb{N}_0 \}$. On $\mathcal{P}, \langle . , . \rangle = ( . , . )_{L^2}$ is positive definite.

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\begin{align*}
&i) \langle \chi_i , v_i \rangle = (v_i , v_i) = 1 \quad ii) (v_i , v_j) = \langle \chi_i , v_j \rangle = 0, \ i \neq j
\end{align*}
Separation and Decomposition

Set $\mathcal{P} \overset{\text{def}}{=} \{ f \in \mathcal{L}_{\alpha}^\beta \mid f^{(k)}(0) = 0, \forall k \in \mathbb{N}_0 \}$. On $\mathcal{P}$, $\langle . , . \rangle = ( . , . )_{L^2}$ is positive definite.

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- ii) $(v_i , v_j) = \langle \chi_i , v_j \rangle = 0, i \neq j$
- iii) $\langle v_i , v_i \rangle = 0$
Separation and Decomposition

Set $\mathcal{P} \overset{\text{def}}{=} \{ f \in \mathcal{S}_\alpha^\beta \mid f^{(k)}(0) = 0, \forall k \in \mathbb{N}_0 \}$. On $\mathcal{P}$, $\langle . , . \rangle = (.,.)_{L^2}$ is positive definite.

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- $\langle \chi_i , v_i \rangle = (v_i , v_i) = 1$
- $\langle v_i , v_j \rangle = 0$, $i \neq j$
- $\langle v_i , v_i \rangle = 0$
- $\langle v_i , f \rangle = f^i$, for all $f \in \mathcal{S}_\alpha^\beta$. 
Separation and Decomposition

Set $\mathcal{P} \overset{\text{def}}{=} \{ f \in \mathcal{L}_\alpha^\beta \mid f^{(k)}(0) = 0, \forall k \in \mathbb{N}_0 \}$. On $\mathcal{P}$, $\langle . , . \rangle = (.,.)_{L^2}$ is positive definite.

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\begin{align*}
  i) \quad & \langle \chi_i , v_i \rangle = (v_i , v_i) = 1 \\
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\end{align*}

The $v_i$ span a closed Hilbert subspace $\overline{\mathcal{N}}^\tau(\ast)$ of $\mathcal{K}$, isomorphic to the $(.,.)$–dual of $\overline{\mathcal{N}}^\tau$. 

\begin{align*}
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Separation and Decomposition

Set $\mathcal{P} \triangleq \{ f \in \mathcal{S}^\beta_\alpha \mid f^{(k)}(0) = 0, \forall k \in \mathbb{N}_0 \}$. On $\mathcal{P}$, $\langle ., . \rangle = (.,.)_{L^2}$ is positive definite.

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**Lemma.** The functionals $F_i(f) \triangleq \langle \chi_i, f \rangle$, $f \in \mathcal{S}^\beta_\alpha$, are $\| . \|_1$–bounded, and thus possess unique continuations to $\mathcal{K}$. The vector representatives $v_i \in \mathcal{K}$, $F_i(.) = (v_i, .)$ are contained in $\overline{\mathcal{P}}^\tau$. They satisfy

\begin{align*}
i) \langle \chi_i, v_i \rangle = (v_i, v_i) = 1 \\
ii) (v_i, v_j) = \langle \chi_i, v_j \rangle = 0, i \neq j \\
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\end{align*}

The $v_i$ span a closed Hilbert subspace $\overline{\mathcal{N}}^\tau(\ast)$ of $\mathcal{K}$, isomorphic to the $\langle ., . \rangle$–dual of $\overline{\mathcal{N}}^\tau$.

**Lemma.** Set $\mathcal{H} \triangleq \overline{\mathcal{P}}^{\tau+}$ where the topology $\tau_+$ is induced by the quadratic Hilbert norm $p_+(.)^2 \triangleq \langle ., . \rangle$ on $\mathcal{P}$. 
Separation and Decomposition

Set $\mathcal{P} \overset{\text{def}}{=} \{ f \in \mathcal{L}_\alpha^\beta \mid f^{(k)}(0) = 0, \forall k \in \mathbb{N}_0 \}$. On $\mathcal{P}$, $\langle . , . \rangle = ( . , . )_{L^2}$ is positive definite.

**Lemma.** The mapping $P : \mathcal{L}_\alpha^\beta \longrightarrow \mathcal{K}$; $f \mapsto f^+$ is continuous in the topology $\tau$, has a continuous extension to $\mathcal{K}$, and is an $( . , . )$–orthogonal projection onto $\overline{\mathcal{P}^\tau}$, which induces the decomposition $\mathcal{K} = \overline{\mathcal{P}^\tau} \oplus \overline{\mathcal{N}^\tau}$.

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**Lemma.** The functionals $F_i(f) \overset{\text{def}}{=} \langle \chi_i , f \rangle$, $f \in \mathcal{L}_\alpha^\beta$, are $\| . \|$–bounded, and thus possess unique continuations to $\mathcal{K}$. The vector representatives $v_i \in \mathcal{K}$, $F_i(.) = (v_i , .)$ are contained in $\overline{\mathcal{P}^\tau}$. They satisfy

\begin{align*}
&i) \langle \chi_i , v_i \rangle = (v_i , v_i) = 1 \\
&ii) (v_i , v_j) = \langle \chi_i , v_j \rangle = 0, i \neq j \\
&iii) \langle v_i , v_i \rangle = 0 \\
&iv) \langle v_i , f \rangle = f^i, \text{ for all } f \in \mathcal{L}_\alpha^\beta.
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**Lemma.** Set $\mathcal{H} \overset{\text{def}}{=} \overline{\mathcal{P}^\tau}$ where the topology $\tau_+$ is induced by the quadratic Hilbert norm $p_+(.)^2 \overset{\text{def}}{=} \langle . , . \rangle$ on $\mathcal{P}$. Then holds the $( . , . )$-orthogonal decomposition

$$\mathcal{K} = \mathcal{H} \oplus \overline{\mathcal{N}^\tau(\ast)} \oplus \overline{\mathcal{N}^\tau}.$$
Gram Operator and Physical Subspace

Let $f$ be infra-exponential of order $\frac{1}{2} = (2^{\frac{1}{2}})$ with strictly positive Taylor coefficients. Then, $(\langle \cdot, \cdot \rangle)$ has a maximal completion to a Krein space $K$. The maximal positive definite subspace of $K$, invariant under the action of the Heisenberg algebra in the Schrödinger representation by selfadjoint operators on it, is $H = F_{L^2(\mathbb{R})}$, where $F_{L^2(\mathbb{R})} = \left\{ f \in L^2(\mathbb{R}) \mid \int_{\mathbb{R}} x^k f(x) dx = 0 \right\}$; $k \geq 0$, and the $k$th moment $\int_{\mathbb{R}} x^k f(x) dx$ of a function $f \in L^2(\mathbb{R})$ is given by $R x^k f(x) dx$. 

Infinite Infrared Regularization in Krein Spaces – p.13/15
\[ \tau \text{ majorant } \overset{\text{Prop. B}}{\Rightarrow} \text{Gram Operator } \langle . , . \rangle = ( . , J . ) \text{ exists.} \]
Gram Operator and Physical Subspace

$\tau$ majorant $\Rightarrow$ Gram Operator $\langle . , . \rangle = ( . , J . )$ exists.

$\langle f , \chi_i \rangle = ( f , J \chi_i ) = ( f , v_i )$, $\forall f \in \mathcal{K}$ $\Rightarrow$ $J \chi_i = v_i$ (by definition of $v_i$)
Gram Operator and Physical Subspace

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\[ \langle f , v_i \rangle = ( f , J v_i ) = \overline{f^i} = ( f , \chi_i ) , \ \forall f \in \mathcal{K} \implies J v_i = \chi_i \quad \text{(by iv) above)} \]
\( \tau \) majorant \( \text{Prop. B} \) Gram Operator \( \langle . , . \rangle = ( . , J . ) \) exists.

\[
\langle f , \chi_i \rangle = ( f , J \chi_i ) = ( f , v_i ) , \forall f \in \mathcal{K} \implies J \chi_i = v_i \quad \text{(by definition of } v_i \text{)}
\]

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\langle f , v_i \rangle = ( f , J v_i ) = \overline{f^i} = ( f , \chi_i ) , \forall f \in \mathcal{K} \implies J v_i = \chi_i \quad \text{(by iv) above)}
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(and \( J|_{\mathcal{H}} = 1 \) \( \implies J = J^{-1} \) is bounded \( \text{Def. C \& Lemma D} \) \( \implies \mathcal{K} \) is a Krein space)
Gram Operator and Physical Subspace

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What is \( \mathcal{H} \)?
Gram Operator and Physical Subspace

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$\langle f , v_i \rangle = ( f , Jv_i ) = f^i = ( f , \chi_i )$, $\forall f \in \mathcal{K}$ $\implies Jv_i = \chi_i$ (by iv) above)

(and $J|_\mathcal{H} = 1$) $\implies J = J^{-1}$ is bounded $^{\text{Def. C & Lemma D}}$ $\mathcal{K}$ is a Krein space

What is $\mathcal{H}$?

$$i^k \hat{f}^{(k)} (0) = \left( i^k \frac{d^k}{d\xi^k} \int_{\mathbb{R}} e^{-i\xi x} f(x) dx \right) \bigg|_{\xi=0} = i^k \int_{\mathbb{R}} (-ix)^k f(x) dx = \mu^k (f)$$
Gram Operator and Physical Subspace

$\tau$ majorant $\Rightarrow$ Gram Operator $\langle \cdot , \cdot \rangle = (\cdot , J \cdot)$ exists.

$$\langle f , \chi_i \rangle = (f , J \chi_i) = (f , v_i), \forall f \in \mathcal{K} \implies J \chi_i = v_i \quad \text{(by definition of } v_i)$$

$$\langle f , v_i \rangle = (f , J v_i) = \overline{f^i} = (f , \chi_i), \forall f \in \mathcal{K} \implies J v_i = \chi_i \quad \text{(by iv) above)}$$

(and $J |_{\mathcal{H}} = 1$) $\implies J = J^{-1}$ is bounded $\Rightarrow \mathcal{K}$ is a Krein space

What is $\mathcal{H}$?

$$i^k \hat{f}^{(k)}(0) = \left. \left( i^k \frac{d^k}{d\xi^k} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx \right) \right|_{\xi = 0} = i^k \int_{\mathbb{R}} (-ix)^k f(x) dx = \mu_k(f)$$

$\mathcal{F} : \mathcal{S}_\alpha^\beta \longrightarrow \mathcal{S}_\alpha^\beta$ is an $L^2$-isometry $\Rightarrow$
Gram Operator and Physical Subspace

\( \tau \) majorant \( \overset{\text{Prop. B}}{\Rightarrow} \) Gram Operator \( \langle . , . \rangle = ( . , J . ) \) exists.

\[
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What is \( \mathcal{H} \)?

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i^k \hat{f}^{(k)}(0) = \left. \left( i^k \frac{d^k}{d\xi^k} \int_{\mathbb{R}} e^{-ix\xi} f(x)dx \right) \right|_{\xi=0} = i^k \int_{\mathbb{R}} (-ix)^k f(x)dx = \mu^k(f)
\]

\[
\mathcal{F} : \mathcal{S}_\beta^\alpha \longrightarrow \mathcal{S}_\alpha^\beta \text{ is an } L^2\text{-isometry} \implies \quad \text{Theorem. Let } 0 \leq \alpha \leq \infty, 1 < \beta < \infty, \text{ and } J \text{ be infra-exponential of order } 1/(2\delta) \text{ with strictly positive Taylor coefficients. Then, } (\mathcal{S}_\beta^\alpha, \langle . , . \rangle) \text{ has a maximal completion to a Krein space } \mathcal{K}. \text{ The maximal positive definite subspace of } \mathcal{K}, \text{ invariant under the action of the Heisenberg algebra in the Schrödinger representation by selfadjoint operators on it, is } \mathcal{H} = \mathcal{F} L^2_0(\mathbb{R}), \text{ where } L^2_0(\mathbb{R}) \overset{\text{def}}{=} \{ f \in L^2(\mathbb{R}) \mid \mu^k(f) = 0, \forall k \in \mathbb{N}_0 \}, \text{ where the } k\text{th moment } \mu^k(f) \text{ of a function } f \in L^2(\mathbb{R}) \text{ is given by } \mu^k(f) \overset{\text{def}}{=} \int_{\mathbb{R}} x^k f(x)dx.\]
Sufficient Conditions for Regularization

Let $V$ be a space with non-degenerate, indefinite inner product $\langle \cdot, \cdot \rangle$. Then, $K = V$ as constructed above, is a Krein space with rank of negativity $\# \{ e_i \neq 0 \}$. In particular, setting $x_i \equiv e_i$ and $v_i = e_i$, one has

$$p(v_i)^2 = \lim_{n \to 1} \frac{1}{n} \sum_{i=0}^{n} \langle v_n + \sum_{j=0}^{n} x_i, e_i \rangle^2 = \frac{1}{2}.$$
Sufficient Conditions for Regularization

Theorem. Let $\mathcal{V}$ be a space with non-degenerate, indefinite inner product $\langle \cdot , \cdot \rangle$. 
Sufficient Conditions for Regularization

**Theorem.** Let \( \mathcal{V} \) be a space with non-degenerate, indefinite inner product \( \langle \cdot, \cdot \rangle \).

0) There exists an orthogonal system \( \{ \widetilde{\chi}_i \}_{i \in \mathbb{N}_0} \) of mutually linearly independent, neutral vectors in \( \mathcal{V} \).
Theorem. Let $\mathcal{V}$ be a space with non-degenerate, indefinite inner product $\langle . , . \rangle$.

0) There exists an orthogonal system $\{\tilde{\chi}_i\}_{i \in \mathbb{N}_0}$ of mutually linearly independent, neutral vectors in $\mathcal{V}$.

1) For all $v \in \mathcal{V}$, the unique decomposition $v = \tilde{v}^N + \sum_{i=0}^{N} \tilde{v}^i \tilde{\chi}_i$, $\tilde{v}^i \in \mathbb{C}$, $N \in \mathbb{N}_0$,

becomes asymptotically positive in the sense that $0 \leq \lim_{N \to \infty} \langle \tilde{v}^N + , \tilde{v}^N + \rangle$. 
**Theorem.** Let $\mathcal{V}$ be a space with non-degenerate, indefinite inner product $\langle . , . \rangle$.

0) There exists an orthogonal system $\{\widetilde{X}_i\}_{i \in \mathbb{N}_0}$ of mutually linearly independent, neutral vectors in $\mathcal{V}$.

1) For all $v \in \mathcal{V}$, the unique decomposition $v = \tilde{v}^N + \sum_{i=0}^{N} \tilde{v}^i \tilde{X}_i$, $\tilde{v}^i \in \mathbb{C}$, $N \in \mathbb{N}_0$, becomes asymptotically positive in the sense that $0 \leq \lim_{N \to \infty} \langle \tilde{v}^N, \tilde{v}^N \rangle$.

2) There exists a sequence of complex numbers $\{\gamma_i\}_{i \in \mathbb{N}_0}$ such that both sequences $\{\gamma_i \langle \tilde{X}_i, v \rangle\}$ and $\{\tilde{v}^i / \gamma_i\}$ are in $l^2(\mathbb{N}_0)$. 


Sufficient Conditions for Regularization

**Theorem.** Let $\mathcal{V}$ be a space with non-degenerate, indefinite inner product $\langle . , . \rangle$.

0) There exists an orthogonal system $\{\tilde{\chi}_i\}_{i \in \mathbb{N}_0}$ of mutually linearly independent, neutral vectors in $\mathcal{V}$.

1) For all $v \in \mathcal{V}$, the unique decomposition $v = \tilde{v}^N + \sum_{i=0}^{N} \tilde{v}^i \tilde{\chi}_i$, $\tilde{v}^i \in \mathbb{C}$, $N \in \mathbb{N}_0$, becomes asymptotically positive in the sense that $0 \leq \lim_{N \to \infty} \langle \tilde{v}^N, \tilde{v}^N \rangle$.

2) There exists a sequence of complex numbers $\{\gamma_i\}_{i \in \mathbb{N}_0}$ such that both sequences $\{\gamma_i \langle \tilde{\chi}_i, v \rangle\}$ and $\{\tilde{v}^i / \gamma_i\}$ are in $l^2(\mathbb{N}_0)$.

Then, $\mathcal{K} = \overline{\mathcal{V}}^\tau$ as constructed above, is a Krein space with rank of negativity $\#\{\tilde{\chi}_i \neq 0\}$. 
Theorem. Let $\mathcal{V}$ be a space with non-degenerate, indefinite inner product $\langle \cdot , \cdot \rangle$.

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1) For all $v \in \mathcal{V}$, the unique decomposition $v = \tilde{v}^N + \sum_{i=0}^{N} \tilde{v}^i \tilde{\chi}_i$, $\tilde{v}^i \in \mathbb{C}$, $N \in \mathbb{N}_0$, becomes asymptotically positive in the sense that $0 \leq \lim_{N \to \infty} \langle \tilde{v}^N + \tilde{v}^N, \tilde{v}^N + \rangle$.

2) There exists a sequence of complex numbers $\{\gamma_i\}_{i \in \mathbb{N}_0}$ such that both sequences $\{\gamma_i \langle \tilde{\chi}_i , v \rangle\}$ and $\{\tilde{v}^i / \gamma_i\}$ are in $l^2(\mathbb{N}_0)$.

Then, $\mathcal{K} = \overline{\mathcal{V}}^\top$ as constructed above, is a Krein space with rank of negativity $\#\{\tilde{\chi}_i \neq 0\}$.

In particular, setting $\chi_i \overset{\text{def}}{=} \gamma_i\tilde{\chi}_i$, and $v^i = \tilde{v}^i / \gamma_i$, one has

$$p(v)^2$$
Theorem. Let $\mathcal{V}$ be a space with non-degenerate, indefinite inner product $\langle \cdot , \cdot \rangle$.

0) There exists an orthogonal system $\{\widetilde{\chi}_i\}_{i \in \mathbb{N}_0}$ of mutually linearly independent, neutral vectors in $\mathcal{V}$.

1) For all $v \in \mathcal{V}$, the unique decomposition $v = \tilde{v}^N + \sum_{i=0}^{N} \tilde{v}^i \tilde{\chi}_i$, $\tilde{v}^i \in \mathbb{C}$, $N \in \mathbb{N}_0$, becomes asymptotically positive in the sense that $0 \leq \lim_{N \to \infty} \langle \tilde{v}^N , \tilde{v}^N \rangle$.

2) There exists a sequence of complex numbers $\{\gamma_i\}_{i \in \mathbb{N}_0}$ such that both sequences $\{\gamma_i \langle \tilde{\chi}_i , v \rangle\}$ and $\{\tilde{v}^i / \gamma_i\}$ are in $l^2(\mathbb{N}_0)$.

Then, $\mathcal{K} = \overline{\mathcal{V}}^\tau$ as constructed above, is a Krein space with rank of negativity $\#\{\tilde{\chi}_i \neq 0\}$.

In particular, setting $\chi_i \overset{\text{def}}{=} \gamma_i \tilde{\chi}_i$, and $v^i = \tilde{v}^i / \gamma_i$, one has

$$p(v)^2 \overset{\text{def}}{=} \lim_{N \to \infty} \left[ \langle v^N , f^N \rangle + \sum_{i=0}^{N} \left\{ |\langle v , \chi_i \rangle|^2 + |v^i|^2 \right\} \right]$$
**Sufficient Conditions for Regularization**

**Theorem.** Let \( \mathcal{V} \) be a space with non-degenerate, indefinite inner product \( \langle . , . \rangle \).

0) There exists an orthogonal system \( \{ \tilde{\chi}_i \}_{i \in \mathbb{N}_0} \) of mutually linearly independent, neutral vectors in \( \mathcal{V} \).

1) For all \( v \in \mathcal{V} \), the unique decomposition

\[
 v = \tilde{v}^N + \sum_{i=0}^{N} \tilde{v}^i \tilde{\chi}_i, \quad \tilde{v}^i \in \mathbb{C}, \; N \in \mathbb{N}_0,
\]

becomes asymptotically positive in the sense that

\[
 0 \leq \lim_{N \to \infty} \langle \tilde{v}^N, \tilde{v}^N \rangle.
\]

2) There exists a sequence of complex numbers \( \{ \gamma_i \}_{i \in \mathbb{N}_0} \) such that both sequences \( \{ \gamma_i \langle \tilde{\chi}_i , v \rangle \} \) and \( \{ \tilde{v}^i / \gamma_i \} \) are in \( l^2(\mathbb{N}_0) \).

Then, \( \mathcal{K} = \overline{\mathcal{V}}^\top \) as constructed above, is a Krein space with rank of negativity \( \# \{ \tilde{\chi}_i \neq 0 \} \).

In particular, setting \( \chi_i \overset{\text{def}}{=} \gamma_i \tilde{\chi}_i \), and \( v^i = \tilde{v}^i / \gamma_i \), one has

\[
 p(v)^2 = \lim_{N \to \infty} \left[ \langle v^N, f^N \rangle + \sum_{i=0}^{N} \left\{ |\gamma_i \langle v, \tilde{\chi}_i \rangle|^2 + |\tilde{v}^i / \gamma_i|^2 \right\} \right]
\]
**Theorem.** Let $\mathcal{V}$ be a space with non-degenerate, indefinite inner product $\langle \cdot, \cdot \rangle$.

0) There exists an orthogonal system $\{\tilde{\chi}_i\}_{i \in \mathbb{N}_0}$ of mutually linearly independent, neutral vectors in $\mathcal{V}$.

1) For all $v \in \mathcal{V}$, the unique decomposition $v = \tilde{v}^N + \sum_{i=0}^{N} \tilde{v}^i \tilde{\chi}_i$, $\tilde{v}^i \in \mathbb{C}$, $N \in \mathbb{N}_0$, becomes asymptotically positive in the sense that $0 \leq \lim_{N \to \infty} \langle \tilde{v}^N +, \tilde{v}^N + \rangle$.

2) There exists a sequence of complex numbers $\{\gamma_i\}_{i \in \mathbb{N}_0}$ such that both sequences $\{\gamma_i \langle \tilde{\chi}_i, v \rangle\}$ and $\{\tilde{v}^i / \gamma_i\}$ are in $l^2(\mathbb{N}_0)$.

Then, $\mathcal{K} = \mathcal{V}^T$ as constructed above, is a Krein space with rank of negativity $\#\{\tilde{\chi}_i \neq 0\}$.

In particular, setting $\chi_i \overset{\text{def}}{=} \gamma_i \tilde{\chi}_i$, and $v^i = \tilde{v}^i / \gamma_i$, one has

$$p(v)^2 = \lim_{N \to \infty} \left[ \langle v^N +, f^N + \rangle + \sum_{i=0}^{N} \left\{ |\gamma_i \langle v, \tilde{\chi}_i \rangle|^2 + |\tilde{v}^i / \gamma_i|^2 \right\} \right]^{2) < \infty}.$$
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